Tuesday, March 24, 2015
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Cosmology I - FRW

Lot's and lot's of jargon!!

We have obtained/studied solutions to Einstein's equations w/ and w/out sources. In each case we relied heavily on symmetries to simplify the story. This usually meant idealizing to a single isolated source (Schwarzchild, Kerr) or exploring asymptotic behavior (waves). There is yet another method to get Einstein's eqns to simplify and that is by applying them to the entire universe and "smooth" over non-uniformities. This is the starting point of cosmology. To completely connect this to what we see today, i.e. ~ Kerr, we need to study the complex intermediate problem of structure formation (which we will not do in this course).

When we apply $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ to our universe we must

- Use $T_{\mu\nu} \neq 0$ (we actually have several possible contributions)
- Not assume t-independence (the universe is evolving w/ time)
- Identify what symmetries are present (this will play a significant role in choosing coordinates)

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Symmetries

In our universe (smoothing over local structure) we observe:

Spatial Homogeneity - translation invariance (consider a cylinder: homo but not iso)
Spatial Isotropy - rotation invariance (consider Schwarzchild : iso but not homo)

Note: Not spacetime!

Together these imply that at any point we see rotational invariance. There is no "center" of our universe!

Moreover: Iso + Homo = maximally symmetric spatial geometry (will be very useful a bit later)

Coordinates

We will adopt what are called "comoving coordinates" to describe our spacetime. The coordinates are adapted to the rest frame of the source, even if it expands or shrinks w/ time. That is, if the proper distance between two objects increases (because of spacetime expansion) then the coordinate separation will remain fixed. Recall two objects sensing a gravity wave!

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Putting all of this together we can begin w/

$$ds^2 = -dt^2 + \underbrace{R^2(t)}_{\text{dimensionful}} \underbrace{\overbrace{\gamma_{ij}(u)\, du^i du^j}^{\text{dimensionless}}}_{\text{The } t\text{-independent comoving spatial geometry}} \qquad i,j = 1,2,3$$

(also referred to as $d\sigma^2$)

Note: This metric will preserve whatever spatial symmetry we impose.

An observer using coordinates adapted to a different reference frame (even constant velocity) will see a different metric w/ different symmetries. For example on Earth we notice a dipole anisotropy in the CMB due to the motion of Earth relative to the overall rest frame of the universe.

Since our spatial geometry is maximally symmetric:

$$R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \Rightarrow R_{jl} = \gamma^{ki} R_{ijkl} = 2k\gamma_{jl} \Rightarrow R = \gamma^{lj} R_{jl} = 6k$$

We thus categorize the spatial geometry by the sign of k:

$k=0$	$d\sigma^2 = d\chi^2 + \chi^2 d\Omega_2^2$	$\mathbb{R}^3$	"flat"
$k>0$	$d\sigma^2 = d\chi^2 + \sin^2\chi\, d\Omega_2^2$	S^3	"closed"
$k<0$	$d\sigma^2 = d\chi^2 + \sinh^2\chi\, d\Omega_2^2$	H^3	"open"

Note: $\bar{r}$ and k are dimensionless

Define $d\chi = \dfrac{d\bar{r}}{\sqrt{1-k\bar{r}^2}}$ then

$$d\sigma^2 = \frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega_2^2 \quad \text{w/ } k = 0, \pm 1$$

$k=-1 \quad d\chi = \dfrac{d\bar{r}}{\sqrt{1+\bar{r}^2}} \Rightarrow \chi = \sinh^{-1}(\bar{r})$

$k=0 \quad d\chi = d\bar{r} \Rightarrow \chi = \bar{r}$

$k=+1 \quad d\chi = \dfrac{d\bar{r}}{\sqrt{1-\bar{r}^2}} \Rightarrow \chi = \sin^{-1}(\bar{r})$

In terms of $\bar{r}$ we have:

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega_2^2\right]$$

The <u>Robertson-Walker</u> metric (describes spatially homo + iso w/ t-dep)

Typically one defines:

$a(t) \equiv \dfrac{R(t)}{R_0}$ dimensionless scale-factor

$r \equiv R_0 \bar{r}$ dimensionful distance

$K \equiv \dfrac{k}{R_0^2}$ dimensionful spatial curvature

Then:

$$ds^2 = -dt^2 + \underbrace{a^2(t)\left[\frac{dr^2}{1-Kr^2}\right.}_{\text{unknowns still to be determined from Einstein's equations}} \left. + r^2 d\Omega_2^2\right]$$

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Perfect Fluid Source:

Assume we are at rest w.r.t. overall motion of fluid

$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$ w/ $U^\mu = (1, 0, 0, 0)$

$$= \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} p & \\ 0 & & & \end{pmatrix}$$

Note: $g_{ij} \neq \gamma_{ij}$ since g_{ij} includes $a(t)$

Einstein's equations in trace-reversed form: $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$

If we use the RW-metric and perfect fluid $T_{\mu\nu}$ these become:

$00: \quad -3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p)$

$ij: \quad \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2} = 4\pi G(\rho - p)$

Combining we obtain: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$ The "Friedmann equation"

Solutions are called FRW cosmologies. These models are best suited to our universe.

More jargon: $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ Hubble parameter $= \begin{cases} > 0 & \text{expanding} \\ < 0 & \text{contracting} \end{cases}$

Using $H(t)$ in Friedmann's equation: $\boxed{H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$ We'll use this extensively later.

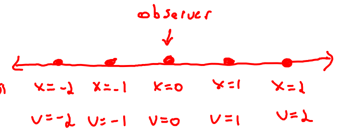
Additionally: $v_{physical} = \frac{d\,\ell_{physical}}{dt} = \frac{d\left[a(t)\,\ell_{coord}\right]}{dt} = \dot{a}\underbrace{\ell_{coord}}_{\text{constant}} = \frac{\dot{a}}{a}\ell_{physical}$

then

$v_{physical} = H\,\ell_{physical}$ "Hubble's Law"

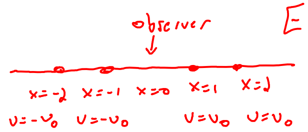
This is an incredibly important result as it describes relative motion in expanding spacetime as distinct from "explosive" expansion in a static spacetime.

Consider:



1D space expanding w/ $H = 1\ s^{-1}$

compared to



Explosive expansion in static space

Note: "No Center" = homogeneity "We are the Center" - no homogeneity

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$H = \frac{\dot a}{a}$ (current value H_0 is measurable)

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

$a(t), \rho(t), K$ ← measurable

ρ_{tot} (discuss various types below)

Since a depends on t and ρ depends on t, we can express how ρ varies w/ a.

Consider: $\nabla_\mu T^{\mu\nu} = 0$ conservation of energy-momentum

or: $g_{\alpha\nu} \nabla_\mu T^{\mu\nu} = \nabla_\mu T^\mu{}_\alpha = 0 \Rightarrow \alpha = 0 \quad \nabla_\mu T^\mu{}_0 = -\frac{d\rho}{dt} - 3\frac{\dot a}{a}(\rho + p) = 0$

Now let's assume that ρ and p are related by an equation of state of the form $p = w\rho$ (constant)

Then:

$$0 = -\dot\rho - 3\frac{\dot a}{a}(1+w)\rho \Rightarrow \frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a} \Rightarrow \ln\rho = -3(1+w)\ln a$$

or

$$\rho(t) \propto a(t)^{-3(1+w)}$$

Consider a few cases:

- Matter (or dust) $\Rightarrow p_m = 0 \Rightarrow w = 0 \Rightarrow \boxed{\rho_m(t) \propto a^{-3}}$ Expresses dilution of fixed particle number due to volume expansion.

- Radiation (or highly relativistic matter) $\Rightarrow p_R = \frac{1}{3}\rho_R \Rightarrow w = \frac{1}{3} \Rightarrow \boxed{\rho_R(t) \propto a^{-4}}$ Volume dilution a^{-3} w/ redshift a^{-1}!

- Vacuum ($T_{\mu\nu} \propto g_{\mu\nu}$) $\Rightarrow p_v = -\rho_v \Rightarrow w = -1 \Rightarrow \boxed{\rho_v(t) \propto a^0 = \text{constant}}$

Then $\rho_{tot} = \rho_m + \rho_R + \rho_v$, but if we define $\underbrace{\rho_c}_{\text{curvature}} \equiv -\frac{3K}{8\pi G a^2}$ then Friedmann's eqn. becomes:

$\rho_c \propto a^{-2}$

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i \qquad i = m, R, v, c$$